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On Rossby waves modified by basic shear, and barotropic instability

By P. G. DRAZIN, D. N. BEAUMONT† AND S. A. COAKER
School of Mathematics, University of Bristol

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The eigenvalue problem of Kuo governing the linear stability of a parallel zonal flow of an inviscid incompressible fluid on a β-plane is treated in this paper. First, a synthesis of the known properties of the normal modes is presented, as a short summary. The rest of the paper is a study of the properties of one of the classes of stable modes, namely Rossby waves modified by the basic shear. These modes are found as the solution of the inverse of a regular Sturm–Liouville problem. Several asymptotic results for small and for increasing values of the basic shear (i.e. equivalently for large and decreasing values of β) are found for quite general velocity profiles. These are illustrated by some numerical calculations of the wave characteristics for a few particular basic velocity profiles.

1. Introduction

In 1939 Rossby devised the β-plane approximation for motion of a thin layer of fluid on a rotating sphere, and thence discovered the waves that now bear his name. It is less well known that in this classic paper he also recognized the nature of barotropic instability, and found the linearized vorticity equation that governs the instability. Ten years later Kuo developed the work on barotropic instability, posing and treating the appropriate eigenvalue problem. The topics of Rossby waves and of barotropic instability have since been extensively treated in the literature, but almost as if they were entirely separate topics. We shall emphasize that the shear of a basic flow modifies the Rossby waves substantially, and, if it is strong enough, makes them become modes of instability. The development and substantiation of this is the chief aim of this paper. Nonetheless, the topics of Rossby waves and barotropic instability are fundamental to so much of modern thought in meteorology and oceanography that it is useful to summarize and synthesize the known results before extending them. The non-specialist reader may find the essential old and new results here and in §7.

In Kuo's (1949) classic model of linear barotropic instability, a basic zonal stream of an inviscid incompressible fluid with velocity $\mathbf{U} = U(y)\,\mathbf{i}$ is supposed to flow between rigid walls, at latitudes $y = y_1$ and y_2, say. Here we take Cartesian coordinates with unit vector $\mathbf{i}$ parallel to the x-axis. Also we take $y_1 < y_2$, and may take $y_1 = -\infty$, $y_2 = \infty$, or both. Then a normal mode of zonal wavenumber α may be shown to be governed by the following eigenvalue problem:

$$\phi'' + \left\{\frac{\beta - U''}{U-c} - \alpha^2\right\}\phi = 0; \tag{1}$$

$$\phi = 0 \quad (y = y_1, y_2). \tag{2}$$

† Present address: British Gas, London Research Station, Michael Road, London SW6 2AD.

The derivative β of the Coriolis parameter will be taken as a constant. If y_1 or y_2 is infinite, we replace (2) by the condition that

$$\phi \text{ is bounded as } y \to y_1 \text{ or } y_2. \tag{3}$$

It is assumed here that a two-dimensional small perturbation has streamfunction $\psi'(x,y,t) = \text{Re}\{\phi(y)\, e^{i\alpha(x-ct)}\}$, where the complex phase velocity $c = c_r + ic_i$ is to be found as an eigenvalue so that the mode is unstable if and only if $\alpha c_i > 0$. The flow is unstable if at least one mode is unstable.

Equation (1) is equivalent to the vorticity equation for the disturbance found by Rossby (1939), but is called the Kuo equation, because Kuo (1949) posed the eigenvalue problem, found some of its general properties, and solved it for some particular flows. Others have found further general and particular properties of barotropic instability. Much of this work has been described by Kuo (1973) in a survey. Nonetheless, the new systematic summary below may be helpful in unifying disparate results.

For a given basic state (i.e. for given $U(y), y_1, y_2$ and $\beta > 0$), it is found that each mode of zonal wavenumber α belongs to one of the following eight classes, any or many of which may be empty.

Bound states. These are defined, by analogy with wave mechanics, as modes for which $\int_{y_1}^{y_2} \phi^2 < \infty$, so that either both y_1 and y_2 are finite, or ϕ decays exponentially at infinity like

$$\exp[-\{\alpha^2 - \beta/(U_{\pm\infty} - c)\}^{\frac{1}{2}} |y|],$$

when there exists

$$U_{\pm\infty} = \lim_{y \to \pm\infty} U(y).$$

(i) A finite number of non-singular unstable modes with eigenvalues c belonging to eigenfunctions ϕ. A necessary (but not sufficient) condition for the existence of these barotropically unstable modes is that $U'' = \beta$ somewhere in the domain of flow (Kuo 1949). The eigenvalues lie inside the semicircle with inequalities

$$\{c_r - \tfrac{1}{2}(U_m + U_M)\}^2 + c_i^2 \leqslant \{\tfrac{1}{2}(U_M - U_m)\}^2 + \frac{\beta(U_M - U_m)}{2\alpha^2}, \quad \alpha c_i > 0 \tag{4}$$

in the complex c-plane, where

$$U_m = \min_{y_1 \leqslant y \leqslant y_2} U(y), \quad U_M = \max_{y_1 \leqslant y \leqslant y_2} U(y);$$

also $c_r < U_M$ (Pedlosky 1964, p. 212).

(ii) An equal number of non-singular damped stable modes, whose eigenvalues and eigenfunctions are complex conjugates of corresponding members of the first class of unstable modes.

(iii) A finite number of marginally stable modes. (There are usually none of these because it is unlikely that a given pair of values of α and β lies on a stability boundary.) They are limits of members of the above two classes as $c_i \to 0$. Their eigenvalues satisfy the inequalities $U_m - \beta/2\alpha^2 < c < U_M$. Their eigenfunctions in general have logarithmic singularities at the critical latitudes where $U = c$; if, however, a mode has only one critical latitude then ϕ is non-singular and $U'' = \beta$ at that latitude. One of these modes may exceptionally be a coincident pair of modes of the next class (see also §2).

(iv) A countable number (finite or infinite) of non-singular stable modes (cf. Dikiy & Katayev 1971). To make the classification unique, we require that these modes are not marginally stable. These have $c < U_m$, Pedlosky (1964, p. 211) having shown that there is no non-singular mode with $c > U_M$: their group velocity may, however, lie within the range of U. They exist only when $U'' < \beta$ somewhere in the flow, and may be identified as Rossby waves modified by the basic shear.

(v) A continuum of singular neutrally stable modes with $U_m < c < U_M$. These have eigenfunctions with discontinuous derivatives at critical latitudes where $U = c$. This continuous spectrum is associated with disturbances that grow or decay algebraically as time increases. Little is known about these modes (but see Warn & Warn 1978).

Unbound states. At an infinite 'boundary' one may replace condition (2) by condition (3). Then there exist unbound states, so defined that $\int_{y_1}^{y_2} \phi^2 = \infty$, of physical interest. They occur if and only if (*a*) $y_1 = -\infty$, $y_2 = \infty$ or both; and (*b*) $\beta/(U_{\pm\infty} - c) > \alpha^2$, and so are stable. Then any mode may be resolved as the superposition of solutions of scattering problems, in each of which a wave of unit amplitude is incident from $y = \pm\infty$ and may give rise to a transmitted wave and a reflected wave. There is total reflection sometimes, for example if $\beta/(U_\infty - c) > \alpha^2$ and either y_1 is finite or $\beta/(U_{-\infty} - c) < \alpha^2$.

(vi) A continuum of non-singular neutral modes with $c < U_m$. For these modes wave-action flux is conserved.

(vii) A continuum of singular neutral modes with $U_m < c < U_M$. Wave action of these modes may be generated or absorbed by interaction with the basic flow at the critical latitudes where the eigenfunction is singular. Over-reflection is possible (cf. Lindzen & Tung 1978), i.e. the wave-action flux of the reflected wave may be greater than that of the incident wave.

(viii) A finite number (usually zero) of singular marginally stable modes. These are limits of modes of classes (i) and (ii) as $c_i \to 0$ and of modes of class (vii). They have infinite over-reflection.

The distribution of modes among these classes varies with the parameters α and β as well as the basic flow. So it is important to know the properties of the modes as α and β vary. Increase of β promotes stability in general but not, one presumes, always; certainly a flow is stable if

$$\beta > \max_{y_1 \leqslant y \leqslant y_2} U''(y).$$

Similarly, an increase of α promotes stability of the modes. This paper is chiefly concerned with the properties of modified Rossby waves, i.e. of the neutrally stable modes of class (iv), as α and β vary and is inspired by the work of Dikiy & Kataev (1971) on Rossby–Haurwitz waves on a sphere. We shall see that in general the wave velocity decreases monotonically to minus infinity as β increases, and increases monotonically to U_m as α increases. It may be seen by dimensional analysis that more-appropriate parameters are αL and $\beta L^2/V$, if an overall lengthscale L and velocity scale V of dynamically similar basic flows may be chosen. Thus the limit as $\beta \to 0$ is equivalent to the limit as $V \to \infty$, in which the problem reduces to the Rayleigh stability problem (see e.g. Drazin & Reid 1981, chap. 4). Also in the limit as either $\beta \to \infty$ or $V \to 0$ the problem reduces to that of classic Rossby waves in the absence of a basic flow (see §3).

First, a few general properties of the eigenvalue relation will be presented in §2. The asymptotic properties of the modified waves as $\beta \to \infty$, which is equivalent to the limit as $c \to -\infty$, are found in §3 for general profiles. This gives the first-order correction to a Rossby wave's velocity and spatial structure due to the basic shear. The rest of the paper is a study of the modified waves as β decreases from infinity, i.e. as c increases from minus infinity to U_m. The dispersion relation and the spatial structure of the waves are shown to be substantially modified by the basic shear. The wave characteristics for an unbounded parabolic basic velocity profile and those for a semi-bounded linear basic velocity profile are given respectively in appendices A and B.† These two special problems are of intrinsic interest but seem to be of little direct practical importance. They are, however, important components of the asymptotic solutions for quite general velocity profiles in §§4 and 5 respectively in the limit as $c \uparrow U_m$. Some numerical results for a few special basic flows are presented in §6; these results both illustrate and are interpretable in the light of the preceding general results. Thereby the paper draws a picture of the wave characteristics as a whole.

2. The dispersion relation

Equation (1) and the boundary conditions (2) form a regular Sturm–Liouville problem to determine the eigenvalues β of the modified Rossby waves for any given values of $\alpha^2 \geqslant 0$ and $c < U_m$. It follows that there exist eigenvalues $\beta_1 < \beta_2 < \dots$ belonging to a complete set $\{\phi_n\}$ of eigenfunctions such that ϕ_n has $n-1$ zeros between the boundaries. Further, by the Sturmian theory of oscillations, β_n increases as α^2 increases.

If $\alpha^2 = 0$ and $\beta = 0$, then any solution of (1) may be expressed as

$$\phi = (U-c)\int (U-c)^{-2}\, dy.$$

Now $c < U_m$. Therefore no solution can vanish at both boundaries. This conclusion is valid *a fortiori* if $\alpha^2 > 0$. It follows that $\beta_n > 0$ for all $\alpha^2 \geqslant 0$, for all n, and for all basic flows.

The above demonstrates that β is a single-valued function of α, n and c. However, the physical nature of the problem requires us to find c as a function of α, β and n. To examine this function and see that it is not always single-valued, denote the eigenvalue relation, i.e. the dispersion relation of the waves, by the form

$$F(\alpha^2, \beta, c) = 0,$$

for each value of n. Suppose then that this relation is satisfied by the particular set of values α_0^2, β_0 and c_0 with eigenfunction ϕ_0. Therefore one may plausibly expand

$$\begin{aligned} 0 &= F(\alpha^2, \beta, c) - F(\alpha_0^2, \beta_0, c_0) \\ &= \left[\frac{\partial F}{\partial \alpha^2}\right]_0 (\alpha^2 - \alpha_0^2) + \left[\frac{\partial F}{\partial \beta}\right]_0 (\beta - \beta_0) + \left[\frac{\partial F}{\partial c}\right]_0 (c - c_0) + \dots, \end{aligned}$$

where the subscript zero denotes evaluation at $(\alpha_0^2, \beta_0, c_0)$. Therefore

$$c - c_0 \sim -\frac{\left\{\left[\dfrac{\partial F}{\partial \alpha^2}\right]_0 (\alpha^2 - \alpha_0^2) + \left[\dfrac{\partial F}{\partial \beta}\right]_0 (\beta - \beta_0)\right\}}{\left[\dfrac{\partial F}{\partial c}\right]_0} \quad \text{as} \quad \alpha^2 \to \alpha_0^2, \quad \beta \to \beta_0, \tag{5}$$

† These appendices may be obtained from the authors or from the Editorial Office of the *Journal of Fluid Mechanics*.

provided that $[\partial F/\partial c]_0 \neq 0$; and

$$(c-c_0)^2 \sim -\frac{\left\{\left[\dfrac{\partial F}{\partial \alpha^2}\right]_0(\alpha^2-\alpha_0^2)+\left[\dfrac{\partial F}{\partial \beta}\right]_0(\beta-\beta_0)\right\}}{\dfrac{1}{2}\left[\dfrac{\partial^2 F}{\partial c^2}\right]_0} \quad \text{as} \quad \alpha^2 \to \alpha_0^2, \quad \beta \to \beta_0, \tag{6}$$

provided that $[\partial F/\partial c]_0 = 0$ and $[\partial^2 F/\partial c^2]_0 \neq 0$.

Equation (5) implies that c is a single-valued function of α^2 and β as β decreases from infinity until $[\partial F/\partial c]_0 = 0$ or F ceases to be differentiable, because c is single-valued when β is large (see §3). This onset of a multiplicity of values of c will be exemplified in §6. Further, (6) shows that instability occurs with purely imaginary values of $c-c_0$ near real values of α_0^2, β_0 and c_0 if $[\partial F/\partial c]_0 = 0$. Thus a single-valued c for given n is associated with stability and multi-valued c with the onset of instability. Note that $c < U_m$ at the onset, as found numerically by Delblonde (1981) for the Bickley jet.

The ratios of the partial derivatives of F can be found by use of the solvability condition for $\psi = \phi-\phi_0$. If ϕ and ϕ_0 satisfy the eigenvalue problem (1) and (2) for their respective eigenvalues, then

$$\psi''-\left(\alpha_0^2+\frac{U''-\beta_0}{U-c_0}\right)\psi \sim \left\{(\alpha^2-\alpha_0^2)-\frac{\beta-\beta_0}{U-c_0}+\frac{U''-\beta_0}{(U-c_0)^2}(c-c_0)\right\}\phi_0, \tag{7}$$

$$\psi = 0 \quad (y = y_1, y_2) \tag{8}$$

in the limit as $\alpha^2 \to \alpha_0^2$, $\beta \to \beta_0$ and $c \to c_0$. The operator on the left-hand side of (7) is self-adjoint, so that the solvability condition for ψ is the usual one that the right-hand side is orthogonal to the solution ϕ_0 of the homogenous problem. This gives

$$\int_{y_1}^{y_2}\left\{(\alpha^2-\alpha_0^2)-\frac{\beta-\beta_0}{U-c_0}+\frac{U''-\beta_0}{(U-c_0)^2}(c-c_0)\right\}\phi_0^2\,dy \to 0. \tag{9}$$

Therefore

$$c-c_0 \sim -\frac{\left\{-(\alpha^2-\alpha_0^2)\displaystyle\int_{y_1}^{y_2}\phi_0^2\,dy+(\beta-\beta_0)\int_{y_1}^{y_2}\frac{\phi_0^2}{U-c_0}dy\right\}}{\displaystyle\int_{y_1}^{y_2}\frac{\beta_0-U''}{(U-c_0)^2}\phi_0^2\,dy} \tag{10}$$

as $\alpha^2 \to \alpha_0^2$ and $\beta \to \beta_0$ for fixed n, if

$$\int_{y_1}^{y_2}\frac{\beta_0-U''}{(U-c_0)^2}\phi_0^2\,dy \neq 0,$$

as is found in (5). This gives the group velocity as

$$c_g = \frac{\partial(\alpha c)}{\partial \alpha} = c+2\alpha^2\frac{\partial c}{\partial \alpha^2} = c+\frac{2\alpha^2\displaystyle\int_{y_1}^{y_2}\phi^2\,dy}{\displaystyle\int_{y_1}^{y_2}\frac{\beta-U''}{(U-c)^2}\phi^2\,dy} \tag{11}$$

If
$$\beta > \max_{y_1 \leqslant y \leqslant y_2} U''(y),$$
then
$$\int_{y_1}^{y_2}\frac{\beta-U''}{(U-c)^2}\phi^2\,dy > 0,$$

and we may further deduce that c increases as β decreases and as α^2 increases, and that $c_g > c$.

Also, if $[\partial F/\partial c]_0 = 0$, i.e. if

$$\int_{y_1}^{y_2} \frac{\beta_0 - U''}{(U-c_0)^2}\phi_0^2\, dy = 0,$$

then the Liapounov–Schmidt method (cf. Banks & Drazin 1973, §6) may be used to express (6) in terms of explicit integrals of ϕ_0 etc.

3. Modified Rossby waves for large β

In this section we consider only bounded flows. Then one may take $y_1 = -\pi$ and $y_2 = \pi$ without loss of generality, by making a linear transformation of the variable y if necessary.

Formally letting $\beta \to \infty$ in the eigenvalue problem (1) and (2), one finds that $c/\beta \to$ constant, where

$$\phi'' - (\alpha^2 + \beta/c)\,\phi = 0, \tag{12}$$

$$\phi = 0 \quad (y = \pm\pi). \tag{13}$$

Therefore

$$c = -\beta c_{n0} \equiv \frac{-\beta}{\alpha^2 + \frac{1}{4}n^2}, \quad \phi = \phi_{n0} \equiv \sin\tfrac{1}{2}n(\pi + y) \quad (n = 1, 2, \ldots), \tag{14}$$

say, the eigenfunction ϕ_{n0} having $n-1$ zeros between the rigid boundaries at $y = \pm\pi$. This is the solution for a classic Rossby wave in the absence of a basic flow.

Coaker (1980) has shown that the eigensolution with $n-1$ zeros may be expanded as

$$c = c_n = -\beta c_{n0} + c_{n1} + \beta^{-1}c_{n2} + \ldots), \quad \phi = \phi_n = \phi_{n0} + \beta^{-1}\phi_{n1} + \ldots \tag{15a, b}$$

as $\beta \to \infty$, uniformly in y (but not in α or n). Substitution of these series into the problem (1) and (2) gives (14) at the first approximation. At the next approximation one finds that

$$L_n\phi_{n1} \equiv \phi_{n1}'' + \tfrac{1}{4}n^2\phi_{n1} \tag{16}$$

$$= c_{n0}^{-2}(c_{n0}U'' - c_{n1} + U)\,\phi_{n0}, \tag{17}$$

$$\phi_{n1} = 0 \quad (y = \pm\pi). \tag{18}$$

Much as we deduced the condition (9) from (7) and (8), one can show that the solvability condition of (17) and (18) is that

$$0 = \int_{-\pi}^{\pi} c_{n0}^{-2}(c_{n0}U'' - c_{n1} + U)\,\phi_{n0}^2\, dy.$$

Therefore

$$c_{n1} = \frac{\int_{-\pi}^{\pi}(U + c_{n0}U'')\,\phi_{n0}^2\, dy}{\int_{-\pi}^{\pi}\phi_{n0}^2\, dy} = \frac{1}{\pi}\int_{-\pi}^{\pi}(U + c_{n0}U'')\sin^2\tfrac{1}{2}n(\pi + y)\, dy. \tag{19}$$

This gives the first-order correction to the Rossby-wave velocity due to the basic flow. One can proceed similarly to find ϕ_{n1}, c_{n2} etc. in turn (Coaker 1980).

4. The limit as $c \uparrow U_m$: velocity profiles with a simple minimum

We noted in §2 that the Sturm–Liouville problem (1) and (2) for $-\infty < c < U_m$ and $\alpha^2 \geqslant 0$ has an infinity of eigenvalues $\beta_n > 0$. In §3 we found β_n as $c \downarrow -\infty$. In appendices A and B† we consider the basic flows with an unbounded parabolic profile

$$U(y) = \tfrac{1}{2}U''_{\mathrm{m}}y^2 \quad (-\infty < y < \infty), \tag{20}$$

and with a semibounded linear profile

$$U(y) = U'_{\mathrm{m}}y \quad (0 \leqslant y < \infty)$$

respectively, and use the method of matched asymptotic expansions to find the modified Rossby waves as $c \uparrow U_{\mathrm{m}}$ $(= 0)$. Next we shall use those results to find β_n as $c \uparrow U_{\mathrm{m}}$ for quite general basic velocity profiles. In this latter limit, (1) becomes nearly singular, and the singularity will be seen to dominate the problem.

First, suppose that U has a simple global minimum: then there exists a unique point $y_{\mathrm{m}} \in (y_1, y_2)$ such that $U_{\mathrm{m}} = U(y_{\mathrm{m}})$, $U'_{\mathrm{m}} \equiv U'(y_{\mathrm{m}}) = 0$, $U''_{\mathrm{m}} \equiv U''(y_{\mathrm{m}}) > 0$ and $U_{\mathrm{m}} < U(y)$ if $y \in [y_1, y_2]$ and $y \neq y_{\mathrm{m}}$.

It will be seen that this case is similar to that of the parabolic profile (20) of appendix A in the limit as $c \uparrow U_{\mathrm{m}}$, because the singularity is similar. So we shall use the method of matched asymptotic expansions, borrowing or modifying results of appendix A where appropriate.

In the formal limit as $c \uparrow U_{\mathrm{m}}$, (1) becomes the outer equation

$$\phi''_{\mathrm{o}} + \left(-\alpha^2 + \frac{\beta - U''}{U - U_{\mathrm{m}}}\right)\phi_{\mathrm{o}} = 0, \tag{21}$$

valid as an approximation for fixed $y \in [y_1, y_{\mathrm{m}})$ or $(y_{\mathrm{m}}, y_2]$. It follows from the theory of regular singularities of ordinary differential equations that, if $\nu \equiv (\tfrac{9}{4} - 2\beta/U''_{\mathrm{m}})^{\frac{1}{2}} \neq 0, \tfrac{1}{2}$ then

$$\phi_{\mathrm{o}}(y) \sim \begin{cases} D_+(y-y_{\mathrm{m}})^{\frac{1}{2}+\nu} + E_+(y-y_{\mathrm{m}})^{\frac{1}{2}-\nu} & \text{as} \quad y \downarrow y_{\mathrm{m}}, & (22a) \\ D_-(y_{\mathrm{m}}-y)^{\frac{1}{2}+\nu} + E_-(y_{\mathrm{m}}-y)^{\frac{1}{2}-\nu} & \text{as} \quad y \uparrow y_{\mathrm{m}}, & (22b) \end{cases}$$

for some constants $D_\pm$ and $E_\pm$. The ratio D_-/E_- is determined (in principle) by integration of the outer equation (21) and use of the boundary condition (2) at $y = y_1$. Likewise D_+/E_+ is independently determined by use of (2) at $y = y_2$.

In general the outer equation (21) is singular at y_{m}. If $\nu = \tfrac{1}{2}$ and $U'''(y_{\mathrm{m}}) = 0$, however, then it is regular although relations (22) are still valid. If $\nu = \tfrac{1}{2}$ and $U'''(y_{\mathrm{m}}) \neq 0$, or if $\nu = 0$, then logarithmic terms may arise in these relations.

We need, in general, an inner solution uniformly valid near $y = y_{\mathrm{m}}$ as $c \uparrow U_{\mathrm{m}}$ in order to find the connection formula that relates D_+/E_+ to D_-/E_- and hence determines the limit of the eigenvalue, $\lim_{c \uparrow U_{\mathrm{m}}} \beta_n = B_n$, say. Taking appendix A as a guide, we let $Y = [U''_{\mathrm{m}}/2(U_{\mathrm{m}} - c)]^{\frac{1}{2}}(y - y_{\mathrm{m}})$, transform the independent variable of (1) to Y, let $c \uparrow U_{\mathrm{m}}$ for fixed Y, and deduce (A 17) in the limit. Then the solution (A 18) follows in the present case and may be rewritten equivalently as

$$\phi_{\mathrm{i}}(Y) = (Y^2 + 1)^{\frac{1}{2}}\{BP^1_{-\frac{1}{2}+\nu}(iY) + B^*P^1_{-\frac{1}{2}+\nu}(-iY)\}, \tag{23}$$

in terms of associated Legendre functions.

Consider first the special case when U is an even function, $y_1 = -y_2$ and $y_{\mathrm{m}} = 0$. Then we may use the boundary conditions (A 5) that $\phi' = 0$ or $\phi = 0$ at $y = 0$ and (2) at $y = y_2$, considering the interval $y_1 \leqslant y < 0$ only implicitly by use of the evenness or oddness of the eigenfunctions ϕ_n for odd and even n respectively. Thus it follows that

$$\phi_{\mathrm{i}}(Y) = B(Y^2 + 1)^{\frac{1}{2}}\{P^1_{-\frac{1}{2}+\nu}(iY) \pm P^1_{-\frac{1}{2}+\nu}(-iY)\},$$

† See footnote on p. 442.

respectively. It may now be deduced, similarly to the deduction of asymptotic relation (A 19), that

$$\phi_i(y) \sim \frac{2B}{(-2\pi)^{\frac{1}{2}}} \frac{2^{\nu}\Gamma(\nu)\, Y^{\frac{1}{2}+\nu}}{\Gamma(\nu-\frac{1}{2})} \begin{Bmatrix} \cos\frac{1}{2}\pi(\frac{1}{2}+\nu) \\ i\sin\frac{1}{2}\pi(\frac{1}{2}+\nu) \end{Bmatrix} \quad \text{as} \quad Y\uparrow\infty, \tag{24}$$

respectively, unless either $\nu^2 < 0$ or one of $\cos\frac{1}{2}\pi(\frac{1}{2}+\nu)$, $\sin\frac{1}{2}\pi(\frac{1}{2}+\nu)$ and $1/\Gamma(\nu-\frac{1}{2})$ vanishes. Thus the limiting eigenvalues B_n are determined in general as the eigenvalues (if any) of the outer equation (21) for $0 < y < y_2$ such that $\phi_o = 0$ at $y = y_2$ and $\phi_o \sim \text{constant} \times (y-y_m)^{\frac{1}{2}+\nu}$ as $y\downarrow y_m$, i.e. such that $E_+/D_+ \to 0$ as $c\uparrow U_m$.

Further, on matching to the next approximation, we find that

$$\phi_o(y) = \frac{2B}{(-2\pi)^{\frac{1}{2}}} \frac{2^{\nu_n}\Gamma(\nu_n)}{\Gamma(\nu_n-\frac{1}{2})} \left\{\frac{U''_m}{2(U_m-c)}\right\}^{\frac{1}{2}(\frac{1}{2}+\nu_n)} \begin{Bmatrix} \cos\frac{1}{2}\pi(\frac{1}{2}+\nu_n) \\ i\sin\frac{1}{2}\pi(\frac{1}{2}+\nu_n) \end{Bmatrix}$$
$$\times [\{1+(\beta-B_n)\, D_{+1}\}(y-y_m)^{\frac{1}{2}+\nu_n} + (\beta-B_n)\, E_{+1}(y-y_m)^{\frac{1}{2}-\nu_n} + O\{(\beta-B_n)^2\}]$$

respectively as $y\downarrow y_m$, $\beta\to B_n$, where $\nu_n = (\frac{9}{4}-2B_n/U''_m)^{\frac{1}{2}}$. Here we have used the matching relation

$$\lim_{Y\uparrow\infty} \phi_i(Y) \sim \lim_{y\downarrow y_m} \phi_o(y) \quad \text{as} \quad c\uparrow U_m$$

to determine D_+ in terms of B to the first approximation and assumed that D_+ and E_+ have Taylor series in powers of $\beta-B_n$. The coefficients of the terms proportional to $\beta-B_n$ are the constants D_{+1} and E_{+1} after normalization, which we may regard as being known in principle on integration of the outer problem. Now, matching the coefficients of $(y-y_m)^{\frac{1}{2}-\nu_n}$ in this expression of ϕ_o with the expression (24) for ϕ_i, we deduce that in general

$$\frac{2B}{(-2\pi)^{\frac{1}{2}}} \frac{2^{\nu_n}\Gamma(\nu_n)}{\Gamma(\nu_n-\frac{1}{2})} \left\{\frac{U''_m}{2(U_m-c_n)}\right\}^{\frac{1}{2}(\frac{1}{2}+\nu_n)} \begin{Bmatrix} \cos\frac{1}{2}\pi(\frac{1}{2}+\nu_n) \\ i\sin\frac{1}{2}\pi(\frac{1}{2}+\nu_n) \end{Bmatrix} (\beta-B_n)\, E_{+1}$$
$$\sim \frac{2B}{(-2\pi)^{\frac{1}{2}}} \frac{2^{-\nu_n}\Gamma(-\nu_n)}{\Gamma(-\nu_n-\frac{1}{2})} \left\{\frac{U''_m}{2(U_m-c_n)}\right\}^{\frac{1}{2}(\frac{1}{2}-\nu_n)} \begin{Bmatrix} \cos\frac{1}{2}\pi(\frac{1}{2}-\nu_n) \\ i\sin\frac{1}{2}\pi(\frac{1}{2}-\nu_n) \end{Bmatrix} \quad \text{as} \quad \beta\to B_n.$$

Therefore
$$U_m - c_n \sim a_n|\beta-B_n|^{1/\nu_n} \quad \text{as} \quad \beta\to B_n < \tfrac{9}{8}U''_m \tag{25}$$

for $n = 1, 2, \ldots, p$, where a_n are some positive constants, which depend on the profile and α^2.

The number p of these modes is finite because the problem has only a finite number of eigenvalues $B_n < \frac{9}{8}U''_m$ to ensure that ν_n is real. This number may be zero and indeed $p = 0$ for the unbounded parabolic profile of appendix A. The integer p depends not only on the profile but also on α^2, and is a monotone decreasing function of α^2.

In addition, there exists for a general profile of this section an infinity of modes similar to those found for an unbounded parabolic profile. Their structure and deduction follows much as in appendix A. It follows that

$$c_n - U_m \sim -b_n e^{-(n+1-q)\pi/\mu} \quad \text{as} \quad \beta\downarrow\tfrac{9}{8}U''_m \tag{26}$$

for $n > q$, where b_n are some positive constants, which depend on the profile and α^2, and q is some integer not less than p.

Various special cases for which the general argument leading to (25) is invalid have been ignored hitherto: when $\nu_n = 0$ or $\frac{1}{2}$, when E_{+1} happens to vanish, when $\cos\frac{1}{2}\pi(\frac{1}{2}\pm\nu_n)$, $\sin\frac{1}{2}\pi(\frac{1}{2}\pm\nu_n)$, $1/\Gamma(\pm\nu_n-\frac{1}{2})$ or $1/\Gamma(-\nu_n)$ happen to vanish. It is lengthy to analyse each of the special cases, so it seems best to summarize them by noting that we have found one possibility in addition to the asymptotic relations (25) and (26): namely that

$$c_n - U_{\mathrm{m}} \sim d_n(\beta - U''_{\mathrm{m}})^{\rho} \quad \text{as} \quad \beta \to U''_{\mathrm{m}}, \tag{27}$$

for some integer n and constants d_n, $\rho > 0$ in the special case when $B_n = U''_{\mathrm{m}}$ and therefore $\nu_n = \frac{1}{2}$.

Also note that we have, for the sake of simplicity and brevity, presented the arguments above for even profiles U. The arguments and their conclusions are similar, however, for general profiles with a simple global minimum. Then the outer problem for each value of β determines the ratios D_+/E_+ and D_-/E_-. These ratios are connected by matching to the inner solution, which may be continued from $Y = \infty$ to $Y = -\infty$ to determine first B_n and then the asymptotic eigenvalue relation in the form (25). Likewise, (26) also follows for general profiles.

Lastly, we conjecture that, as for the unbounded parabolic flow of appendix A, (26) is also valid as $n \to \infty$ or as $\alpha^2 \to \infty$ for fixed β and a general profile.

5. The limit as $c \uparrow U_{\mathrm{m}}$: velocity profiles with a minimum at a wall

Here we suppose that U attains its global minimum at a wall and that the basic shear does not vanish at the wall: then either $U'(y_1) > 0$ and $U(y) > U(y_1)$ if $y \in (y_1, y_2]$; or $U'(y_2) < 0$ and $U(y) > U(y_2)$ if $y \in [y_1, y_2)$. The two alternatives are essentially equivalent, so we may suppose without loss of generality that $U'_{\mathrm{m}} \equiv U'(y_1) > 0$ and $U(y) > U(y_1) \equiv U_{\mathrm{m}}$ if $y \in (y_1, y_2]$.

To solve the eigenvalue problem in the limit as $c \uparrow U_{\mathrm{m}}$ we follow Beaumont (1980). To understand the argument it may again help to bear in mind that the thin critical layer near the wall is the dominant feature of the flow and therefore that the general case resembles the semibounded linear profile of appendix B.

Putting $c = U_{\mathrm{m}}$ formally, we find the outer equation,

$$\phi''_{\mathrm{o}} + \left\{-\alpha^2 + \frac{\beta - U''}{U - U_{\mathrm{m}}}\right\}\phi_{\mathrm{o}} = 0, \tag{28}$$

and the outer boundary condition that

$$\phi_{\mathrm{o}} = 0 \quad (y = y_2). \tag{29}$$

This outer problem is singular at $y = y_1$. The theory of singularities of linear ordinary equations, however, gives

$$\phi_{\mathrm{o}}(y) \sim D(y - y_1) + E(y - y_1)\ln(y - y_1) \quad \text{as} \quad y \downarrow y_1,$$

for some constants D and E whose ratio can be found (in principle) by integration of (28) after use of the initial condition (29).

The exact equation (1) is, however, not singular at y_1. So its inner solution valid near $y = y_1$ that satisfies the boundary condition (2) at $y = y_1$ is of the form

$$\phi_{\mathrm{i}}(y) = F(y - y_1) \tag{30}$$

for some constant F. This conclusion can be deduced more formally by letting $c \uparrow U_{\mathrm{m}}$ for fixed inner variable $Y \equiv U'_{\mathrm{m}}(y - y_1)/(U_{\mathrm{m}} - c)$.

In any event, matching

$$\lim_{Y \uparrow \infty} \phi_{\mathrm{i}}(Y) \sim \lim_{y \downarrow y_1} \phi_{\mathrm{o}}(y) \quad \text{as} \quad c \uparrow U_{\mathrm{m}},$$

we deduce that $D \to F$ and $E \to 0$. It follows that the limiting eigenvalue

$$B_n = \lim_{c \uparrow U_m} \beta_n$$

is determined by the outer problem together only with the condition that ϕ_0 vanishes *linearly* as $y \downarrow y_1$.

6. Examples

Next we present some numerical and analytical results concerning Rossby waves modified by a few particular basic flows. These examples are chosen more to illustrate the above results for general profiles than for their own sakes. Thereby an overall picture of the characteristics of modified Rossby waves will be drawn.

6.1. *Bounded parabolic profile*

First, we follow Coaker (1980) in considering the basic velocity profile with

$$U(y) = \left(\frac{y}{\pi}\right)^2 \quad (-\pi \leqslant y \leqslant \pi). \tag{31}$$

This flow is stable for all values of β, by Kuo's generalization of Rayleigh's inflexion-point criterion. Note that the distribution U is even and $y_1 = -y_2$, so that each discrete eigenfunction is either even or odd; in particular, ϕ_n is even if n is odd and is odd if n is even. On evaluation of the integral (19), (15*a*) gives

$$c_n = -\frac{\beta}{\alpha^2+\frac{1}{4}n^2} + \frac{2}{\pi^2}\left\{\frac{1}{\alpha^2+\frac{1}{4}n^2} - \frac{1}{n^2}\right\} + \tfrac{1}{3} + O(\beta^{-1}) \quad \text{as} \quad \beta \to \infty. \tag{32}$$

The profile has a simple global minimum at $y_m = 0$, where

$$U_m = 0, \quad U''_m = 2/\pi^2 = 0{\cdot}2026, \quad U'''_m = 0 \quad \text{and} \quad U^{iv}_m = 0.$$

Extensive numerical calculations have been performed, agreeing convincingly with asymptotic results (32) as $c \downarrow -\infty$ and (25)–(27) as $c \uparrow U_m$. The calculations are summarized in figure 1. Coaker (1980) gives some further details, in particular some graphs of $\phi_n(y)$.

Figure 1 (*a*) indicates that the asymptotic formula (32) is quite accurate, even when β is not large, and that its accuracy increases as n decreases for fixed β. Calculations of c_1 for several values of α and β indicates that the accuracy of (32) increases as α^2 decreases for fixed n. Indeed, when $\alpha^2 = 0$, (32) is very accurate for all values of β; (32) gives $\beta \to 0{\cdot}2353$ as $c_1 \uparrow 0$ instead of the 'exact' result $\beta = B_1 = U''_m = 0{\cdot}2026$. It seems that $B_1 = U''_m$ for α^2 at least as high as 10 but not for $\alpha^2 = 100$. When α^2 is sufficiently large the JWKB approximation suggests that the solution is, except in thin layers near the walls, as if the flow were unbounded, and therefore that the results of appendix A are applicable. It follows that the modes are of type (26) with $B_n = \frac{9}{8}U''_m$ if α^2 is sufficiently large. Figure 1 (*b*) confirms that c_n increases towards U_m as α^2 increases.

It seems that there is no mode of type (25) with $B_n \neq U''_m$ for any values of n and α^2.

6.2. *Bounded sinusoidal profile*

Coaker (1980) also considered the basic velocity profile with

$$U(y) = \sin y \quad (-\pi \leqslant y \leqslant \pi). \tag{33}$$

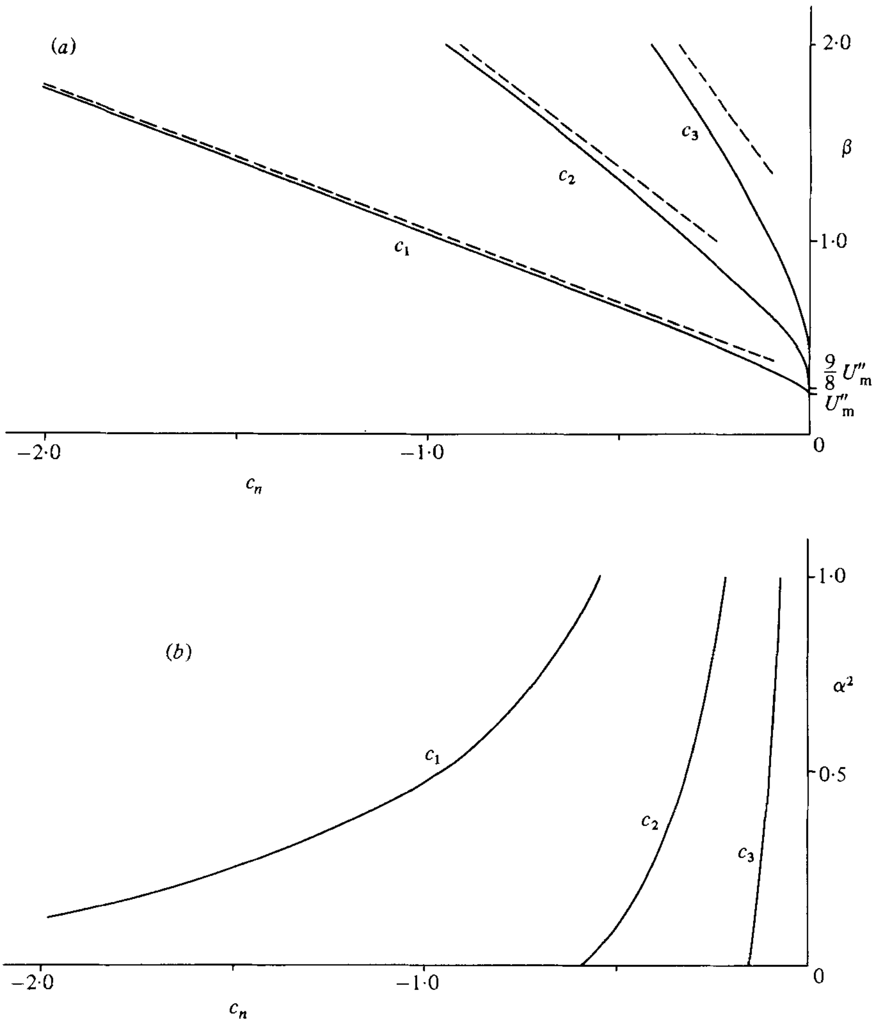

FIGURE 1. $U(y) = (y/\pi)^2$ for $-\pi \leqslant y \leqslant \pi$. (*a*) The continuous curves give β *vs.* c_n for $\alpha^2 = 0{\cdot}5$ and $n = 1, 2, 3$. The broken lines give $c_n = -\beta/(\alpha^2+\frac{1}{4}n^2)+2\pi^{-2}\{(\alpha^2+\frac{1}{4}n^2)^{-1}-n^{-2}\}+\frac{1}{3}$, i.e. the second-order approximation for large β. Here $c_1(\beta)$ is of type (25) with $B_1 = U''_m = 2\pi^{-2} = 0{\cdot}2026$, and c_2 and c_3 are of type (26) with $B_n = \frac{9}{8}U''_m$ as $c_n \uparrow U_m$. (*b*) The continuous curves give α^2 *vs.* c_n for $\beta = 1$ and $n = 1, 2, 3$.

This flow is certainly stable for all $\beta > \max U'' = 1$. It can be seen by inspection that two eigensolutions of the system (1) and (2) are given by

$$c_1 = -\beta, \quad \phi_1(y) = \cos\tfrac{1}{2}y \quad (\alpha^2 = \tfrac{3}{4}), \tag{34a}$$

$$c_2 = -\beta, \quad \phi_2(y) = \sin y \quad (\alpha^2 = 0), \tag{34b}$$

for all β. Equations (19) and (15*a*) give

$$c_n = -\frac{\beta}{\alpha^2+\frac{1}{4}n^2}+O\left(\frac{1}{\beta}\right) \quad \text{as} \quad \beta \to \infty \quad \text{for fixed } n, \tag{35}$$

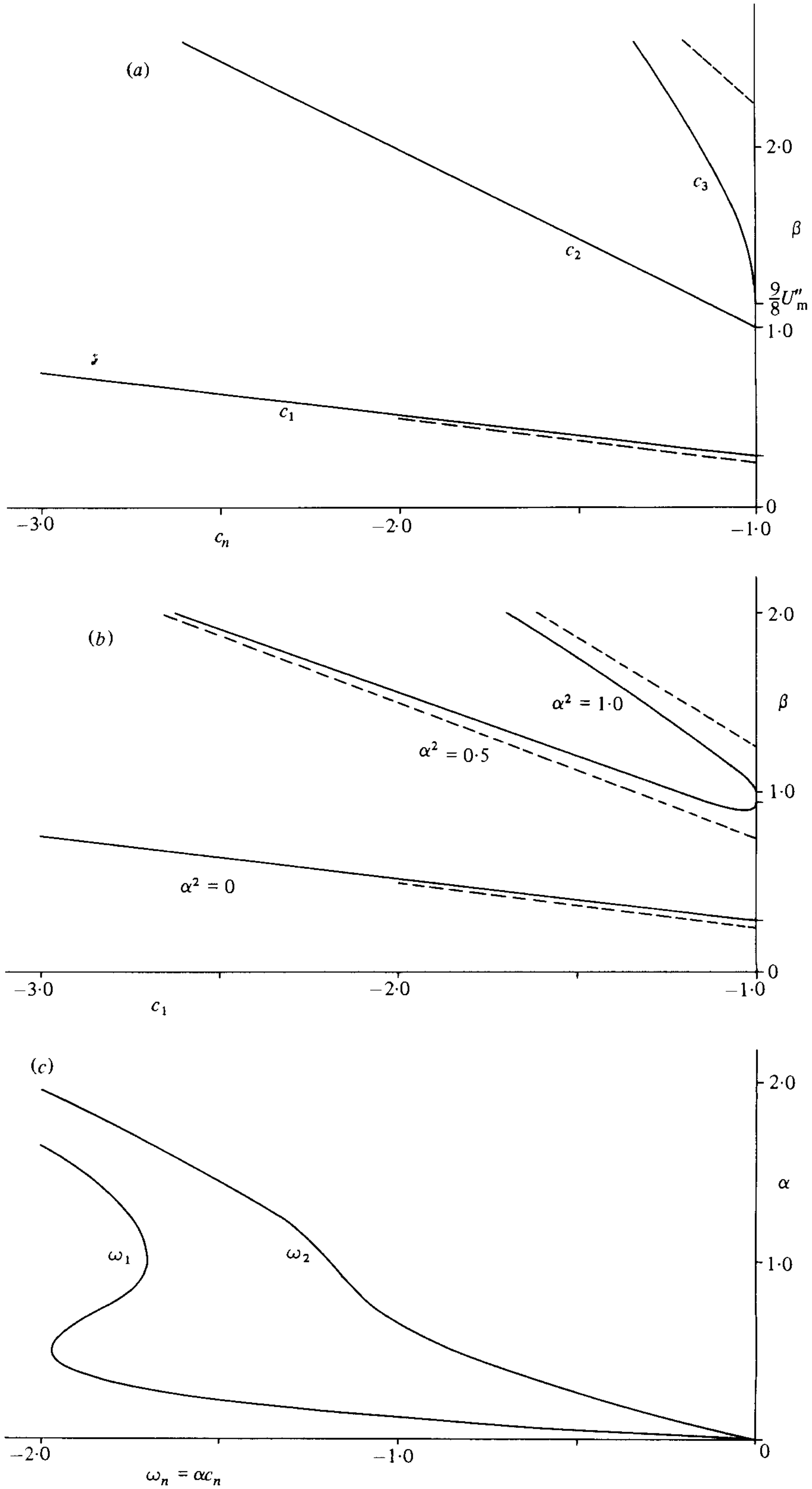

FIGURE 2. For caption see facing page.

all the coefficients of even powers of β in the expansion vanishing because the problem is antisymmetric about $y = 0$. There is a simple global minimum at $y_{\mathrm{m}} = -\frac{1}{2}\pi$, where $U_{\mathrm{m}} = -1$, $U''_{\mathrm{m}} = 1$, $U'''_{\mathrm{m}} = 0$ and $U^{\mathrm{iv}}_{\mathrm{m}} = -1$.

Again, extensive numerical calculations agree well with the analytical and asymptotic results. The calculations are summarized in figure 2, further details being reported by Coaker (1980). The results are qualitatively quite similar to those for the bounded parabolic profile. Figure 2(*a*) gives an example, however, of a mode of type (25) as $c_1 \uparrow U_{\mathrm{m}}$ with $B_1 < U''_{\mathrm{m}}$. Note that the formula (35) for large $-\beta$ happens to give the exact result (34) when $n = 2$ and $\alpha^2 = 0$ (or $n = 1$ and $\alpha^2 = \frac{3}{4}$). Figure 2(*b*) gives $c_1(\beta)$ for various values of α^2. Note that, when $\alpha^2 = 0{\cdot}5$, c_1 is not a single-valued function of $\beta < 1$ and the flow is presumably unstable. Figure 2(*c*) gives the dispersion relation for $\beta = 2$; it can be seen that the group velocity vanishes and therefore lies within the range of U; further, ω_1 is a single-valued function of α but α is not a single-valued function of ω_1 for all ω_1.

6.3. *An asymmetric channel flow*

Beaumont (1980) chose the basic velocity profile with

$$U(y) = \frac{27}{32}\left\{\left(\frac{y}{\pi}\right)^3 + \left(\frac{y}{\pi}\right)^2 - \left(\frac{y}{\pi}\right) - 1\right\} \quad (-\pi \leqslant y \leqslant \pi) \tag{36}$$

in order to illustrate the general results for asymptotic profiles. Again extensive calculations have been made for this profile, and results found qualitatively similar to those for the sinusoidal flow (33) and so serve to confirm that the qualitative results do not depend upon any special symmetry of the basic flow. Beaumont (1980) gives some details of the numerical results.

6.4. *The Bickley jet*

Beaumont (1980) also considered the basic profile,

$$U(y) = -\operatorname{sech}^2 y \quad (-\infty < y < \infty), \tag{37}$$

in order to illustrate the wave characteristics of unbounded jets. This jet is in fact stable if and only if $\beta \geqslant \max U'' = 2$ (cf. Kuo 1973, §VIIB). Note again that the symmetry of the problem about $y = 0$ implies that each non-singular eigenfunction is either an even or an odd function of y. The boundary conditions (2) require that $\alpha^2 + \beta/c < 0$ in order that the eigenfunction does not grow exponentially at infinity; therefore $-\beta/\alpha^2 < c$. Also $c < U_{\mathrm{m}}$, as always. There is a simple global minimum at $y_{\mathrm{m}} = 0$, where $U_{\mathrm{m}} = -1$, $U''_{\mathrm{m}} = 2$, $U'''_{\mathrm{m}} = 0$ and $U^{\mathrm{iv}}_{\mathrm{m}} \neq 0$.

Calculations reveal that the solution

$$c = -1, \quad \beta = \tfrac{1}{9}\alpha^2(9-\alpha^2), \quad \phi = (\operatorname{sech} y)^{\frac{1}{3}\alpha^2}|\tanh y|^{2-\frac{1}{3}\alpha^2},$$

found by Howard & Drazin (1964), is a *singular* limit of the regular first mode as

FIGURE 2. $U(y) = \sin y$ for $-\pi \leqslant y \leqslant \pi$. (*a*) The continuous curves give β *vs.* c_n for $\alpha^2 = 0$ and $n = 1, 2, 3$. The broken lines give $c_n = -4\beta/n^2$, i.e. the *second*-order approximation for large β. Here $c_1(\beta)$ is of type (25) with $B_1 = 0{\cdot}2874$, $c_2(\beta)$ is of type (27) with $\rho = 1$, $B_2 = U''_{\mathrm{m}} = 1$, and $c_3(\beta)$ is of type (26) with $B_3 = \frac{9}{8}U''_{\mathrm{m}} = 1{\cdot}125$ as $c \uparrow U_{\mathrm{m}} = -1$. (*b*) The continuous curves give β *vs.* c_1 for $\alpha^2 = 0$, $0{\cdot}5$ and 1. The broken lines give $c_1 = -\beta/(\alpha^2 + \frac{1}{4})$, i.e. the *second*-order approximation for large β. Here $c_1(\beta)$ is of type (25) as $c \uparrow U_{\mathrm{m}}$ with $B_1(0) = 0{\cdot}2874$, $B_1(0{\cdot}5) = 0{\cdot}944$, and of type (27) with $\rho = 1$, $B_1(1) = U''_{\mathrm{m}} = 1$. (*c*) $\omega_n = \alpha c_n$ *vs.* α for $\beta = 2$ and $n = 1, 2$. Here $c_1 = -7{\cdot}96$ and $c_2 \approx -2$ when $\alpha = 0$, and $c_g = 0$ when $\alpha \approx 0{\cdot}49$ or $0{\cdot}99$.

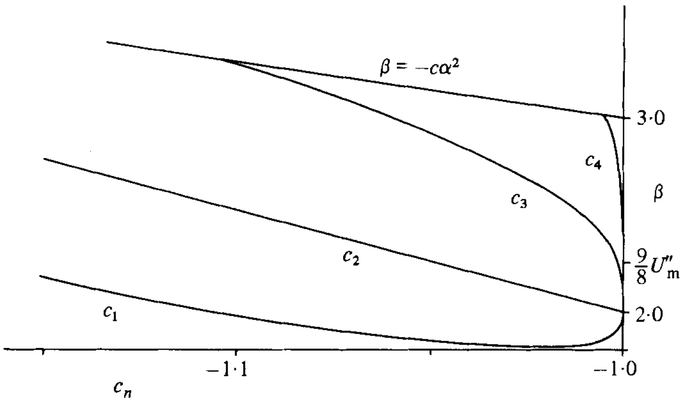

FIGURE 3. $U(y) = -\text{sech}^2 y$ for $-\infty < y < \infty$. β *vs.* c_n for $\alpha^2 = 3$ and $n = 1, 2, 3, 4$. Here it seems that c_1 is of type (25) with $\rho = 4{\cdot}2$, $B_1 = 2$, c_2 is of type (27) with $B_2 = U''_m = 2$, c_3 is of type (27) with $\rho = 3{\cdot}7$, $B_3 = 2$, and c_4 is of type (26) with $B_4 = \frac{9}{8}U''_m = 2{\cdot}25$ as $c_n \uparrow U_m = -1$.

$c_1 \uparrow U_m$ for $0 < \alpha^2 \leqslant 3$. In fact, $B_1(\alpha^2) = 2$ for $3 \leqslant \alpha^2 \leqslant 96$ and perhaps for larger values of α^2 as well.

The calculations made are summarized in figure 3. The chief new point they bring out is how the modified Rossby waves disappear as $c \downarrow -\beta/\alpha^2$. The waves exist then for

$$B_n(\alpha^2) < \beta < B_n^\dagger(\alpha^2),$$

say, where $c_n \downarrow -B_n^\dagger/\alpha^2$ as $\beta \uparrow B_n^\dagger$ with $B_1^\dagger > B_2^\dagger > \dots > -\alpha^2 U_m$. It seems that $B_1^\dagger = \infty$. The calculations give $B_3^\dagger(3) \approx 3{\cdot}32$ and $B_4^\dagger(3) \approx 3{\cdot}02$. A few further details are reported by Beaumont (1980). The argument of §2 indicates that two modified Rossby waves coalesce and lead to instability where the curve c_1 has a minimum in figure 3, because c_1 is a double-valued function of β and α. This supports the work of Delblonde (1981), who re-examined the instability of the Bickley jet and found a modified Rossby mode on the margin of stability at approximately the same values of c_1, α and β.

6.5. *Bounded linear profile*

Beaumont (1980) also considered the basic velocity profile with

$$U(y) = \frac{y}{\pi} \quad (-\pi \leqslant y \leqslant \pi) \tag{38}$$

in order to illustrate the general wave characteristics of profiles with a minimum at a wall. However, this flow is stable for all β. Equations (19) and (15*a*) now give

$$c_n = -\frac{\beta}{\alpha^2 + \frac{1}{4}n^2} + O(\beta^{-1}) \quad \text{as} \quad \beta \to \infty \quad (n = 1, 2, \dots); \tag{39}$$

all the coefficients of even powers of β in this series vanishing because of the antisymmetry of the problem about $y = 0$. Here U has its global minimum at the wall $y_m = -\pi$, where $U_m = -1$ and $U'_m = 1/\pi$.

The exact solution of (1) for the profile (38) in terms of confluent hypergeometric functions is again possible, as in appendix B. Beaumont has given the eigenvalue relation in terms of these functions, but the paucity of tables of them makes the exact eigenvalue relation not of much practical use for general values of α^2. In the particular

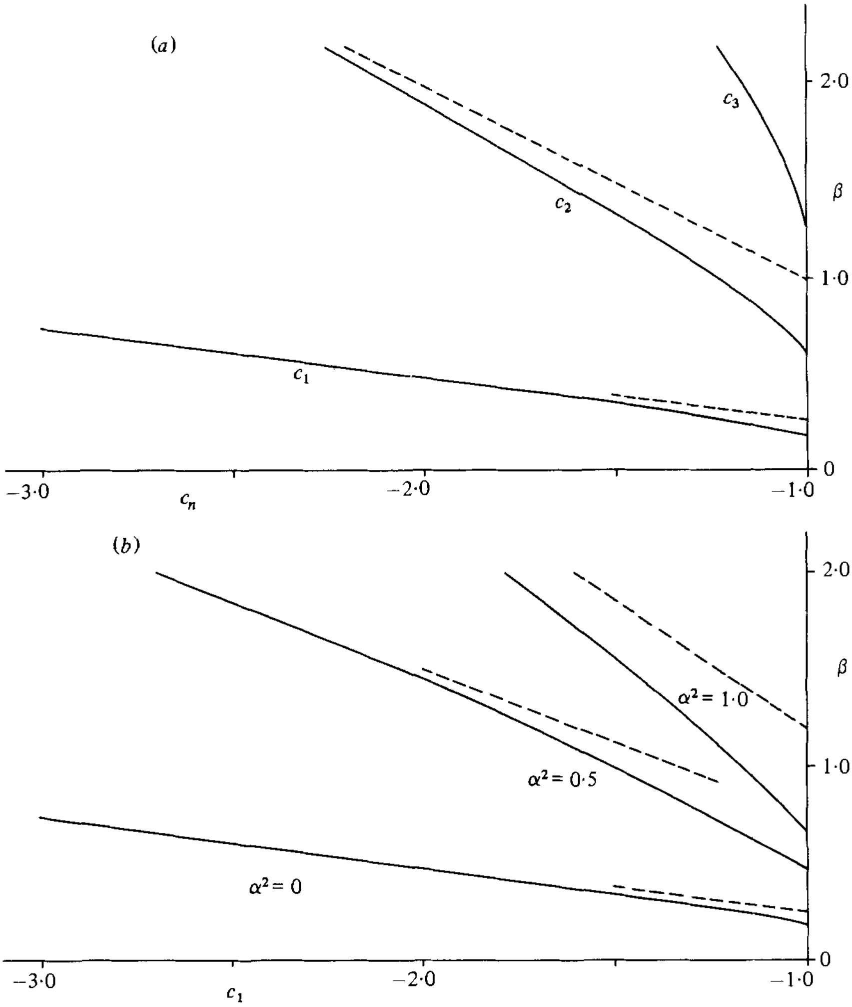

FIGURE 4. $U(y) = y/\pi$ for $-\pi \leqslant y \leqslant \pi$. (a) The continuous curves give β *vs.* c_n for $\alpha^2 = 0$ and $n = 1, 2, 3$. The broken lines give $c_n = -4\beta/n^2$, i.e. the *second*-order approximation for large β. (b) The continuous curves give β *vs.* c_1 for $\alpha^2 = 0$, 0·5 and 1. The broken lines give $c_1 = -\beta/(\alpha^2+\frac{1}{4})$ i.e. the *second*-order approximation for large β.

case when $\alpha^2 = 0$, however, it follows that the general solution of (1) with the profile (38) is

$$\phi(y) = \left(\frac{y}{\pi}-c\right)^{\frac{1}{2}}\left\{AJ_1\left(2\pi\left[\beta\left(\frac{y}{\pi}-c\right)\right]^{\frac{1}{2}}\right)+BY_1\left(2\pi\left[\beta\left(\frac{y}{\pi}-c\right)\right]^{\frac{1}{2}}\right)\right\},$$

where J_1 and Y_1 are the Bessel functions of first order and A and B are arbitrary constants. Beaumont deduced that the eigenvalue relation is

$$J_1(2\pi[\beta(1-c)]^{\frac{1}{2}})\,Y_1(2\pi[\beta(-1-c)]^{\frac{1}{2}})-J_1(2\pi[\beta(-1-c)]^{\frac{1}{2}})\,Y_1(2\pi[\beta(1-c)]^{\frac{1}{2}}) = 0, \quad (40)$$

that it agrees with the relation (39) and also that it gives

$$\beta = \frac{j_{1,n}^2}{8\pi^2}-\frac{j_{1,n}^3}{32\pi}\frac{Y_1(j_{1,n})}{J_1'(j_{1,n})}(-1-c)+O\{(-1-c)^2\} \quad \text{as} \quad c\uparrow -1 \quad (n = 1, 2, \ldots), \quad (41)$$

where $j_{1,n}$ is the nth positive zero of J_1.

Our extensive numerical calculations are summarized in figure 4. It is verified that the asymptotic relations (39) and (41) are accurate in their appropriate domains. Note the difference in the behaviour of β as $c \uparrow U_m$ according to whether the flow has its minimum velocity at a wall or within the domain of flow.

Beaumont (1980) has given further numerical results. He has also briefly analysed two other profiles with minimum velocities at the wall; namely,

$$U(y) = -y^2 \quad (-1 \leqslant y \leqslant 1),$$

$$U(y) = -e^{-y} \quad (0 \leqslant y < \infty).$$

The results seem to be similar qualitatively to those of the linear profile (38).

7. Conclusions

The great detail of the above results makes them difficult for a non-specialist to assimilate. Yet the essence of the results can be quite simply extracted and made readily digestible as follows. The structure of the problem of Rossby waves and barotropic instability as a whole is described briefly in §1, and the problem of modified Rossby waves, in particular, is treated more intensively in §2. The dispersion relation of the waves for general profiles at large values of β is derived in §3. In particular, (15*a*) gives

$$c_n = -\frac{\beta}{\alpha^2+\frac{1}{4}n^2}+O(1) \quad \text{as} \quad \beta \to \infty$$

for $n = 1, 2, \cdots$, for all fixed α^2 and for all basic flows. As β decreases, or rather as c increases, for fixed n and α^2, there are two general types of asymptotic results according to whether the minimum U_m of the basic velocity U occurs within the domain of flow or at a wall.

(i) If the minimum occurs within the domain of flow (and is a simple minimum, i.e. $U''_m > 0$) then (25)–(27) give

(*a*) $U_m - c_n \sim a_n|\beta - B_n|^{1/\nu_n}$ as $\beta \to B_n < \frac{9}{8}U''_m$ $(n = 1, 2, \ldots, p)$;

(*b*) $U_m - c_n \sim d_n(\beta - U''_m)^\rho$ as $\beta \to U''_m$ $(n = p+1, p+2, \cdots, q)$;

(*c*) $U_m - c_n \sim -b_n e^{-(n+1-q)\pi/\mu}$ as $\beta \downarrow \frac{9}{8}U''_m$ $(n = q+1, q+2, \ldots)$

respectively. Here p and $q-p$ are some non-negative integers, and a_n, b_n, d_n, ρ and B_n are some positive constants, which depend upon α^2 as well as the basic flow. Further, the asymptotic relation (*c*) is valid in general as $\alpha^2 \to \infty$ or as $n \to \infty$.

(ii) If the minimum U_m occurs at a wall (and the shear there is non-zero, i.e. $U'_m \neq 0$) then §5 gives

$$U_m - c_n \sim d_n(\beta - B_n) \quad \text{as} \quad \beta \to B_n \quad (n = 1, 2, \ldots).$$

Special cases, not considered here, arise if $U''_m = 0$ at a minimum within the domain of flow or if $U'_m = 0$ at a minimum at a wall.

The examples of §6 may be used to put flesh on to the bones of these asymptotic relations. In fact, one may make a good qualitative sketch of the graphs of c_n against β for various values of α^2 and n using little more than the universal results summarized in the previous paragraph but one. The results as $\beta \to \infty$ are essentially those for classic Rossby waves, because the influence of the basic flow is weak. Another important fact to note is that the results as $c \uparrow U_m$ are essentially the same for all members of each of the two broad classes of basic velocity profiles (i) and (ii), i.e. those whose minimum velocity occurs between the walls and those whose minimum occurs at a wall. This is because the proximity of a singularity to the domain of flow implies that the problem, physically and mathematically, is dominated by the local structure of the disturbances near the latitude where the basic zonal velocity attains its minimum value.

It should not be forgotten that the properties of the modified Rossby waves are of little physical importance when the basic flow is unstable. Thus, for example, the waves calculated in §6.2 would be swamped by barotropic instability or turbulence if β were less than one.

The treatment of the problem of this paper has wider connotations. There are many similar problems with similar properties deducible by similar means (see e.g. Drazin & Howard 1966). In particular, the analogous problem of Rossby–Haurwitz waves on a sphere has been considered by Dikiy & Katayev (1971), some of whose results anticipate ours. Also the work of Bell (1974), Banks, Drazin & Zaturska (1976) and Leibovich (1979) on the analogous problem of internal gravity waves is similar. It is noteworthy, however, that the singularity in the problem of internal gravity waves differs, so that the case of a simple minimum of the velocity profile within the domain of flow is most similar to that of a minimum at a wall in Banks *et al.* Indeed, the two problems, of Rossby waves and of internal gravity waves, are archetypal of the two types of singularities, according to whether terms in $(U-c)^{-1}$ or $(U-c)^{-2}$ respectively arise in the wave equation of perturbations of a parallel shear flow. For these reasons one may conjecture that the methods of the present paper may be useful for solving many other problems, although the results may differ in detail.

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REFERENCES

BANKS, W. H. H. & DRAZIN, P. G. 1973 Perturbation methods in boundary-layer theory. *J. Fluid Mech.* **58**, 763–775.

BANKS, W. H. H., DRAZIN, P. G. & ZATURSKA, M. B. 1976 On the normal modes of parallel flow of inviscid stratified fluid. *J. Fluid Mech.* **75**, 149–171.

BEAUMONT, D. N. 1980 The stability of parallel shear flows. Ph.D. dissertation, University of Bristol.

BELL, T. H. 1974 Effects of shear on the properties of internal gravity wave modes. *Deutsche hydrograph. Z.* **27**, 57–62.

COAKER, S. A. 1980 The barotropic instability problem. Ph.D. dissertation, University of Bristol.

DELBLONDE, G. 1981 Instabilité barotrope du jet de Bickley. M.Sc. dissertation, McGill University.

DIKIY, L. A. & KATAYEV, V. V. 1971 Calculation of the planetary wave spectrum by the Galerkin method. *Izv. Atmos. Ocean. Phys.* **7**, 682–686.

DRAZIN, P. G. & HOWARD, L. N. 1966 Hydrodynamic stability of parallel flow of inviscid fluid. *Adv. Appl. Mech.* **9**, 1–89.

DRAZIN, P. G. & REID, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

HOWARD, L. N. & DRAZIN, P. G. 1964 On instability of parallel flow of inviscid fluid in a rotating system with variable Coriolis parameter. *J. Math. & Phys.* **43**, 83–99.

KUO, H. L. 1949 Dynamic instability of two-dimensional non-divergent flow in a barotropic atmosphere. *J. Met.* **6**, 105–122.

KUO, H. L. 1973 Dynamics of quasigeostrophic flows and instability theory. *Adv. Appl. Mech.* **13**, 247–330.

Leibovich, S. 1979 Waves in parallel or swirling stratified shear flows. *J. Fluid Mech.* **93**, 401–412.

Lindzen, R. S. & Tung, K. K. 1978 Wave overreflection and shear instability. *J. Atmos. Sci.* **35**, 1626–1632.

Pedlosky, J. 1964 The stability of currents in the atmosphere and ocean: part I. *J. Atmos. Sci.* **21**, 201–219.

Rossby, C. G. 1939 Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action. *J. Mar. Res.* **2**, 38–55.

Warn, T. & Warn, H. 1978 The evolution of a nonlinear critical level. *Stud. Appl. Math.* **59**, 37–71.